Induced Norms of the Schur Multiplier Operator*

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ABSTRACT

Fix an n-by-n complex matrix A, and consider the operator $X \mapsto S_A(X) \equiv A \circ X$ on n-by-n complex matrices X, where $A \circ X$ denotes the Schur product of A and X. We show that the induced norm of S_A with respect to the numerical-radius norm is at most one if and only if the matrix A admits a factorization $A = B^*WB$, where W is a contractive matrix and the Euclidean norms of the columns of B are at most one. We give other equivalent characterizations and derive, as a consequence, a formally similar result about the induced norm of S_A with respect to the spectral norm.

1. INTRODUCTION AND RESULT

Let $\mathbb{M}_n$ denote the linear space of n-by-n complex matrices. On $\mathbb{M}_n$, besides the usual *spectral norm*

$$\|A\|_\infty \equiv \sup_x \frac{\|Ax\|}{\|x\|},$$

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we consider the *numerical-radius norm*

$$w(A) \equiv \sup_x \frac{|\langle Ax \mid x \rangle|}{\|x\|^2},$$

where $\langle \cdot | \cdot \rangle$ and $\|\cdot\|$ denote the inner product and the Euclidean norm, respectively. It is easy to see that

$$w(A) \leqslant \|A\|_\infty \leqslant 2w(A) \qquad (A \in \mathbb{M}_n). \tag{1}$$

For $A, B \in \mathbb{M}_n$, denote by $A \circ B$ their *Schur* (or *Hadamard*) *product*, that is, the entrywise product. Then each $A \in \mathbb{M}_n$ gives rise to a linear operator S_A on $\mathbb{M}_n$, called the *Schur multiplier operator*, defined by

$$S_A(X) \equiv A \circ X \qquad (X \in \mathbb{M}_n).$$

The induced norm of S_A with respect to the spectral norm $\|\cdot\|_\infty$ will be denoted by $\|S_A\|_\infty$:

$$\|S_A\|_\infty \equiv \sup_X \frac{\|A \circ X\|_\infty}{\|X\|_\infty},$$

while the induced norm of S_A with respect to the numerical-radius norm will be denoted by $\|S_A\|_w$:

$$\|S_A\|_w \equiv \sup_X \frac{w(A \circ X)}{w(X)}.$$

In the sequel, for a pair of Hermitian matrices A, B, the order relation $A \geqslant B$ means that $A - B$ is positive semidefinite.

It is mentioned in [6] (see p. 110 and p. 116) that Haagerup succeeded in determining $\|S_A\|_\infty$ in the following form.

Haagerup Theorem. *For $A=[a_{ij}]\in \mathbb{M}_n$ the following assertions are mutually equivalent*:

(i) $\|S_A\|_\infty \leqslant 1$.
(ii) *A admits a factorization $A = B^*C$ such that*

$$B^*B\circ I\leqslant I \quad \text{and} \quad C^*C\circ I\leqslant I,$$

where I is the identity (or unit) matrix.
(iii) *There are vectors $x_i, y_i \in \mathbb{C}^n$ $(i=1,2,\dots,n)$ such that $\|x_i\|, \|y_i\|\leqslant 1$ $(i=1,2,\dots,n)$ and*

$$a_{ij}=\langle x_j\,|\,y_i\rangle \qquad (i,j=1,2,\dots,n).$$

(iv) *There are $0\leqslant R_1, R_2\in \mathbb{M}_n$ such that*

$$\begin{bmatrix} R_1 & A \\ A^* & R_2 \end{bmatrix}\geqslant 0, \qquad R_1\circ I\leqslant I, \quad \text{and} \quad R_2\circ I\leqslant I.$$

In the present paper we are going to determine the norm $\|S_A\|_w$, and to derive the Haagerup theorem as a consequence.

Theorem. *For $A=[a_{ij}]\in \mathbb{M}_n$ the following assertions are mutually equivalent*:

$(\mathrm{i})_w$ $\|S_A\|_w\leqslant 1$.
$(\mathrm{ii})_w$ *A admits a factorization $A = B^*WB$ such that*

$$B^*B\circ I\leqslant I \quad \text{and} \quad W^*W\leqslant I.$$

$(\mathrm{iii})_w$ *There are vectors $x_i\in\mathbb{C}^n$ $(i=1,2,\dots,n)$ and a contractive matrix $W\in\mathbb{M}_n$, i.e. $W^*W\leqslant I$, such that $\|x_i\|\leqslant 1$ $(i=1,2,\dots,n)$ and*

$$a_{ij}=\langle Wx_j\,|\,x_i\rangle \qquad (i,j=1,2,\dots,n).$$

$(\mathrm{iv})_w$ *There is $0\leqslant R\in\mathbb{M}_n$ such that*

$$\begin{bmatrix} R & A \\ A^* & R \end{bmatrix}\geqslant 0 \quad \text{and} \quad R\circ I\leqslant I.$$

The authors thank Y. Nakamura for pointing out an erroneous statement of the theorem in the original manuscript. The details will be mentioned in the last part of the paper.

2. SEVERAL LEMMAS

A proof of the theorem is given after a series of lemmas. Given $x \in \mathbb{C}^n$, denote by D_x the diagonal matrix with x on the diagonal.

LEMMA 1. $\|S_A\|_w \leqslant 1$ *if and only if*

$$\|D_x \bar{A} D_x^*\|_{w^*} \leqslant \|x\|^2 \qquad (x \in \mathbb{C}^n),$$

where $\bar{A}$ *is the complex conjugate of* A, *and* $\|\cdot\|_{w^*}$ *denotes the dual norm of* $w(\cdot)$,

$$\|Y\|_{w^*} \equiv \sup_X \frac{|\operatorname{tr}(Y^*X)|}{w(X)} \qquad (Y \in \mathbb{M}_n).$$

Proof. It is easy to see that the adjoint operator of S_A is given by $S_{\bar{A}}$ and the unit ball for the $\|\cdot\|_{w^*}$-norm is the absolute convex hull of matrices of the form $x \otimes x^*$ with $\|x\| = 1$, where $x \otimes x^*$ is the product of the column vector x and its conjugate transpose x^*. Since $\|S_A\|_w = \|S_{\bar{A}}\|_{w^*}$, the assertion follows from the obvious relation

$$S_{\bar{A}}(x \otimes x^*) = D_x \bar{A} D_x^*. \qquad \blacksquare$$

Denote by J_k the k-by-k matrix with all entries equal to one:

$$J_k \equiv \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

LEMMA 2. $\|S_A\|_w = \|S_{A \otimes J_k}\|_w$.

Proof. We treat only the case $k = 2$. The general case $(k > 2)$ can be proved by a similar argument. It is clear that

$$\|S_A\|_w \leqslant \|S_{A\otimes J_2}\|_w.$$

Therefore it remains to show that $\|S_A\|_w \leqslant 1$ implies $\|S_{A\otimes J_2}\|_w \leqslant 1$.

According to Lemma 1 this will follow if it is shown that if

$$\|D_x \bar{A} D_x^*\|_{w^*} \leqslant \|x\|^2 \qquad (x \in \mathbb{C}^n) \tag{2}$$

then

$$\left\|\begin{bmatrix} D_y \bar{A} D_y^* & D_y \bar{A} D_z^* \\ D_z \bar{A} D_y^* & D_z \bar{A} D_z^* \end{bmatrix}\right\|_{w^*} \leqslant \|y\|^2 + \|z\|^2 \qquad (y, z \in \mathbb{C}^n). \tag{3}$$

Assume (2). Given $y, z \in \mathbb{C}^n$, take $u \in \mathbb{C}^n$ such that

$$u \circ \bar{u} = y \circ \bar{y} + z \circ \bar{z}. \tag{4}$$

Then obviously

$$\|u\|^2 = \|y\|^2 + \|z\|^2. \tag{5}$$

We can define, without ambiguity, two diagonal matrices U and V by

$$U = D_y D_u^{-1} \quad \text{and} \quad V = D_z D_u^{-1}. \tag{6}$$

It follows from (4) and (6) that

$$[U^*, V^*]\begin{bmatrix} U \\ V \end{bmatrix} \leqslant I;$$

hence $\begin{bmatrix} U \\ V \end{bmatrix}$ is a contraction from $\mathbb{C}^n$ to $\mathbb{C}^{2n}$. Then for any $X \in \mathbb{M}_{2n}$ we have

$$w\left([U^*, V^*] \cdot X \cdot \begin{bmatrix} U \\ V \end{bmatrix}\right) \leqslant w(X). \tag{7}$$

Now since

$$\begin{bmatrix} D_y\bar{A}D_y^* & D_y\bar{A}D_z^* \\ D_z\bar{A}D_y^* & D_z\bar{A}D_z^* \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \cdot D_u\bar{A}D_u^* \cdot [U^*, V^*],$$

we have by (7)

$$\begin{aligned} \left\| \begin{bmatrix} D_y\bar{A}D_y^* & D_y\bar{A}D_z^* \\ D_z\bar{A}D_y^* & D_z\bar{A}D_z^* \end{bmatrix} \right\|_{w^*} &= \sup_{\substack{X \in \mathbb{M}_{2n} \\ w(X) \leqslant 1}} \left| \operatorname{tr}\left(D_u\bar{A}D_u^* \cdot [U^*, V^*] X \begin{bmatrix} U \\ V \end{bmatrix} \right) \right| \\ &\leqslant \sup_{\substack{Y \in \mathbb{M}_n \\ w(Y) \leqslant 1}} \left| \operatorname{tr}\left(D_u\bar{A}D_u^* \cdot Y \right) \right| \\ &= \| D_u\bar{A}D_u^* \|_{w^*} \\ &\leqslant \|u\|^2 \qquad [\text{by } (2)] \\ &= \|y\|^2 + \|z\|^2 \qquad [\text{by } (5)], \end{aligned}$$

proving (3). ∎

Here we mention two characterizations of a matrix whose numerical radius is at most one. The first one is almost trivial: $w(X) \leqslant 1$ if and only if for any real θ the real (or Hermitian) part of $e^{i\theta}X$ is not greater than I, that is,

$$\operatorname{Re}(e^{i\theta}X) \equiv \tfrac{1}{2}(e^{i\theta}X + e^{-i\theta}X^*) \leqslant I \qquad (0 \leqslant \theta \leqslant 2\pi).$$

The second one is nontrivial and is mentioned as a lemma. See [1] for a proof.

LEMMA 3 (Ando). *For a matrix $X \in \mathbb{M}_n$, $w(X) \leqslant 1$ if and only if there is a Hermitian matrix $Z \in \mathbb{M}_n$ such that*

$$\begin{bmatrix} I + Z & X \\ X^* & I - Z \end{bmatrix} \geqslant 0.$$

Let us recall some notions from the theory of C^*-algebras. See [6] for details. Let $\mathscr{A}, \mathscr{B}$ be C^*-algebras with unit. Let $\mathscr{M}$ be a subspace of $\mathscr{A}$

which contains the unit of $\mathscr{A}$ and is closed under the *-operation. A linear map Φ from $\mathscr{M}$ to $\mathscr{B}$ is said to be *unital* if it maps the unit of $\mathscr{A}$ to the unit of $\mathscr{B}$; it is said to be *positive* if it maps positive elements in $\mathscr{M}$ to positive elements in $\mathscr{B}$. For each $k \geqslant 1$ the map Φ induces a linear map Φ_k from $\mathbb{M}_k(\mathscr{M})$, the space of $\mathscr{M}$-valued k-by-k matrices, to $\mathbb{M}_k(\mathscr{B})$ by

$$\Phi_k([a_{ij}]) \equiv [\Phi(a_{ij})] \quad \text{for } a_{ij} \in \mathscr{M}, \qquad i,j = 1,2,\dots,k.$$

Then Φ is said to be *completely positive* if Φ_k is positive for $k = 1,2,\dots$.

Now let $\mathscr{M}$ denote the subspace of $\mathbb{M}_2(\mathbb{M}_n) = \mathbb{M}_2 \otimes \mathbb{M}_n$ defined by

$$\mathscr{M} \equiv \left\{ \begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix} ; X, Y, Z \in \mathbb{M}_n \text{ and } \lambda \in \mathbb{C} \right\}.$$

Then $\mathscr{M}$ contains the unit of $\mathbb{M}_2(\mathbb{M}_n)$ and is closed under the *-operation.

LEMMA 4. *Suppose that* $\|S_A\|_w \leqslant 1$. *Then the linear map* Φ *from* $\mathscr{M}$ *to* $\mathbb{M}_n$ *defined by*

$$\Phi\left(\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix} \right) = \lambda I + \tfrac{1}{2}\{A \circ X + A^* \circ Y\} \tag{8}$$

is unital and completely positive.

Proof. By Lemma 2 we have

$$\|S_{A \otimes J_k}\|_w = \|S_A\|_w \leqslant 1 \qquad (k = 1,2,\dots). \tag{9}$$

Clearly Φ is unital. First let us prove that Φ is positive. Suppose

$$\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix} \geqslant 0.$$

This positivity implies that $Y = X^*$ and $\lambda I \pm Z \geqslant 0$; hence $\lambda \geqslant 0$. We may assume $\lambda > 0$. Then we have

$$\begin{bmatrix} I + \dfrac{Z}{\lambda} & \dfrac{X}{\lambda} \\ \dfrac{X^*}{\lambda} & I - \dfrac{Z}{\lambda} \end{bmatrix} \geqslant 0;$$

hence by Lemma 3 $w(X/\lambda)\leqslant 1$, that is, $w(X)\leqslant\lambda$. Then the assumption $\|S_A\|_w\leqslant 1$ implies $w(A\circ X)\leqslant\lambda$; hence

$$\Phi\left(\begin{bmatrix}\lambda I+Z & X\\ Y & \lambda I-Z\end{bmatrix}\right)=\lambda I+\mathrm{Re}(A\circ X)\geqslant 0.$$

Next let us prove that the map Φ_k from $\mathbb{M}_k(\mathscr{M})$ to $\mathbb{M}_k(\mathbb{M}_n)$ is positive for $k>1$. Supposing that a $2k$-by-$2k$ block matrix

$$\begin{bmatrix}\lambda_{ij}I+Z_{ij} & X_{ij}\\ Y_{ij} & \lambda_{ij}I-Z_{ij}\end{bmatrix}_{1\leqslant i,j\leqslant k}\geqslant 0, \tag{10}$$

we have to prove that

$$\left[\lambda_{ij}I+\tfrac{1}{2}\{A\circ X_{ij}+A^*\circ Y_{ij}\}\right]_{1\leqslant i,j\leqslant k}\geqslant 0. \tag{11}$$

A suitable permutation of the indices $\{1,2,\ldots,k\}$ will show that (10) is equivalent to

$$\begin{bmatrix}I\otimes[\lambda_{ij}]+[Z_{ij}] & [X_{ij}]\\ [Y_{ij}] & I\otimes[\lambda_{ij}]-[Z_{ij}]\end{bmatrix}\geqslant 0 \tag{12}$$

and (11) means

$$I\otimes[\lambda_{ij}]+\tfrac{1}{2}\{(A\otimes J_k)\circ[X_{ij}]+(A^*\otimes J_k)\circ[Y_{ij}]\}\geqslant 0.$$

As in the first part of the proof, (12) implies that $[Y_{ij}]=[X_{ij}]^*$ and

$$I\otimes[\lambda_{ij}]\geqslant \mathrm{Re}\{e^{i\theta}[X_{ij}]\}\qquad (0\leqslant\theta\leqslant 2\pi). \tag{13}$$

Since $[\lambda_{ij}]\geqslant 0$, there is a unitary matrix $U\in\mathbb{M}_k$ and $\rho_i\geqq 0$ $(i=1,2,\ldots,k)$ such that

$$[\lambda_{ij}]=U^*\cdot\mathrm{diag}(\rho_1,\ldots,\rho_k)\cdot U.$$

Then (13) implies

$$I \otimes \operatorname{diag}(\rho_1, \ldots, \rho_k) \geqslant \operatorname{Re}\{e^{i\theta}(I \otimes U) \cdot [X_{ij}] \cdot (I \otimes U^*)\};$$

hence we have a numerical-radius inequality

$$w\left(\{I \otimes \operatorname{diag}(\rho_1, \ldots, \rho_k)^{-1/2} \cdot U\}[X_{ij}]\{I \otimes U^* \cdot \operatorname{diag}(\rho_1, \ldots, \rho_k)^{-1/2}\}\right) \leqslant 1. \tag{14}$$

Since $\|S_{A \otimes J_k}\|_w \leqslant 1$ by (9), it follows from (14) that

$$I \otimes [\lambda_{ij}] + \operatorname{Re}[A \circ X_{ij}] \geqslant 0,$$

which is equivalent to (11). ∎

We need the following two theorems for the proof of the next lemma. Denote by $\mathscr{B}(\mathscr{H})$ the C^*-algebra of all bounded linear operators on a separable Hilbert space $\mathscr{H}$.

ARVESON THEOREM. *Let $\mathscr{M}$ be a subspace of a C^*-algebra $\mathscr{A}$, which contains the unit of $\mathscr{A}$ and is closed under the $*$-operation, and let Φ be a unital completely positive map from $\mathscr{M}$ to $\mathscr{B}(\mathscr{H})$. Then there exists a completely positive map $\tilde{\Phi}$ from $\mathscr{A}$ to $\mathscr{B}(\mathscr{H})$, extending Φ:*

$$\tilde{\Phi}(a) = \Phi(a) \qquad (a \in \mathscr{M}).$$

See [6, p. 81] for a proof.

STINESPRING THEOREM. *Let $\mathscr{A}$ be a C^*-algebra with unit* 1, *and let Φ be a completely positive map from $\mathscr{A}$ to $\mathscr{B}(\mathscr{H})$. Then there exists a Hilbert space $\mathscr{K}$, a unital $*$-homomorphism π of $\mathscr{A}$ into $\mathscr{B}(\mathscr{K})$, and a bounded linear map V from $\mathscr{H}$ to $\mathscr{K}$ such that $\|\Phi(1)\| = \|V\|^2$ and*

$$\Phi(a) = V^* \pi(a) V \qquad (a \in \mathscr{A}).$$

See [6, p. 43] for a proof.

LEMMA 5. *If $\|S_A\|_w \leqslant 1$, there is a Hilbert space $\mathscr{K}$ and linear maps $\tilde{B}, \tilde{C}$ from $\mathbb{C}^n$ to $\mathscr{K}$ such that*

$$A = \tilde{B}^*\tilde{C} \tag{15}$$

and

$$\tilde{B}^*\tilde{B} = \tilde{C}^*\tilde{C} \quad \text{and} \quad \tilde{B}^*\tilde{B} \circ I \leqslant I. \tag{16}$$

Proof. Since the linear map Φ from $\mathscr{M}$ to $\mathbb{M}_n \simeq \mathscr{B}(\mathbb{C}^n)$, defined by (8), is unital and completely positive by Lemma 4, according to the Arveson theorem and the Stinespring theorem there is a Hilbert space $\mathscr{K}$, a $*$-homomorphism π of the C^*-algebra $\mathbb{M}_2(\mathbb{M}_n)$ into $\mathscr{B}(\mathscr{K})$, and a linear map V from $\mathbb{C}^n$ to $\mathscr{K}$ such that

$$\Phi\left(\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix}\right) = V^* \cdot \pi\left(\begin{bmatrix} \lambda I + Z & X \\ Y & \lambda I - Z \end{bmatrix}\right) \cdot V. \tag{17}$$

Then it follows from (17) that

$$V^* \cdot \pi\left(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\right) \cdot V = \tfrac{1}{2} A \circ X,$$

$$V^* \cdot \pi\left(\begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot V = V^* \cdot \pi\left(\begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}\right) \cdot V,$$

and $V^*V = I$. Let $\{e_j\}$ be the canonical orthonormal basis of $\mathbb{C}^n$. Define $\tilde{B}$ and $\tilde{C}$ by

$$\tilde{B}e_j = \sqrt{\frac{2}{n}} \sum_{p=1}^{n} \pi\left(\begin{bmatrix} E_{pj} & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot Ve_j \qquad (j = 1, \ldots, n) \tag{18}$$

and

$$\tilde{C}e_j = \sqrt{\frac{2}{n}} \sum_{p=1}^{n} \pi\left(\begin{bmatrix} 0 & E_{pj} \\ 0 & 0 \end{bmatrix}\right) \cdot Ve_j \qquad (j = 1, \ldots, n), \tag{19}$$

where $E_{ij} = e_i \otimes e_j^*$. For $i, j = 1, \dots, n$ we have by (18) and (19)

$$\langle \tilde{B}^* \tilde{C} e_j \mid e_i \rangle = \frac{2}{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \left\langle V^* \cdot \pi\left(\begin{bmatrix} E_{ip} & 0 \\ 0 & 0 \end{bmatrix} \right) \pi\left(\begin{bmatrix} 0 & E_{qj} \\ 0 & 0 \end{bmatrix} \right) \cdot V e_j \middle| e_i \right\rangle$$

$$= 2 \left\langle V^* \cdot \pi\left(\begin{bmatrix} 0 & E_{ij} \\ 0 & 0 \end{bmatrix} \right) \cdot V e_j \middle| e_i \right\rangle = a_{ij};$$

hence $\tilde{B}^* \tilde{C} = A$. Further, for $i, j = 1, \dots, n$

$$\langle \tilde{B}^* \tilde{B} e_j \mid e_i \rangle = 2 \left\langle V^* \cdot \pi\left(\begin{bmatrix} E_{ij} & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot V e_j \middle| e_i \right\rangle$$

and

$$\langle \tilde{C}^* \tilde{C} e_j \mid e_i \rangle = 2 \left\langle V^* \cdot \pi\left(\begin{bmatrix} 0 & 0 \\ 0 & E_{ij} \end{bmatrix} \right) \cdot V e_j \middle| e_i \right\rangle;$$

hence $\tilde{B}^* \tilde{B} = \tilde{C}^* \tilde{C}$. Finally

$$\begin{aligned} 2\langle \tilde{B}^* \tilde{B} e_j \mid e_j \rangle &= \langle \tilde{B}^* \tilde{B} e_j \mid e_j \rangle + \langle \tilde{C}^* \tilde{C} e_j \mid e_j \rangle \\ &= 2 \left\langle V^* \cdot \pi\left(\begin{bmatrix} E_{jj} & 0 \\ 0 & E_{jj} \end{bmatrix} \right) \cdot V e_j \middle| e_j \right\rangle \\ &\leqslant 2 \langle V^* V e_j \mid e_j \rangle = 2; \end{aligned}$$

hence $\tilde{B}^* \tilde{B} \circ I \leqslant I$. ■

LEMMA 6. *If* $\|S_A\|_w \leqslant 1$, *there exist* $B, W \in \mathbb{M}_n$ *such that*

$$A = B^* W B$$

and

$$B^* B \circ I \leqslant I \quad \text{and} \quad W^* W \leqslant I.$$

Proof. By Lemma 5 there are linear maps $\tilde{B}, \tilde{C}$ from $\mathbb{C}^n$ to a Hilbert space $\mathscr{K}$ satisfying (15) and (16). Then $\tilde{B}$ and $\tilde{C}$ have the same *modulus*,

$$|\tilde{B}| \equiv (\tilde{B}^*\tilde{B})^{1/2} = (\tilde{C}^*\tilde{C})^{1/2} \equiv |\tilde{C}|.$$

Let $B \equiv |\tilde{B}|$. Then first $B^*B \circ I = \tilde{B}^*\tilde{B} \circ I \leqslant I$. Next there are linear maps U, V from $\mathbb{C}^n$ to $\mathscr{K}$ such that

$$\tilde{B} = UB, \quad U^*U = I \quad \text{and} \quad \tilde{C} = VB, \quad V^*V = I.$$

Let $W \equiv U^*V$. Then we have $W^*W \leqslant I$ and

$$A = \tilde{B}^*\tilde{C} = B^*U^*VB = B^*WB. \qquad \blacksquare$$

LEMMA 7. *If*

$$\begin{bmatrix} R & A \\ A^* & R \end{bmatrix} \geqslant 0$$

for some $0 \leqslant R \in \mathbb{M}_n$ *with* $R \circ I \leqslant I$, *then* $\|S_A\|_w \leqslant 1$.

Proof. Take $X \in \mathbb{M}_n$ with $w(X) \leqslant 1$; then by Lemma 3 there is $Z \in \mathbb{M}_n$ such that

$$\begin{bmatrix} I+Z & X \\ X^* & I-Z \end{bmatrix} \geqslant 0.$$

Then according to the Schur theorem (see [3, p. 458]) that the Schur product of two positive semidefinite matrices is positive semidefinite, we have

$$\begin{bmatrix} R \circ (I+Z) & A \circ X \\ A^* \circ X^* & R \circ (I-Z) \end{bmatrix} \geqslant 0. \tag{20}$$

Since $R \circ I \leqslant I$, it follows from (20) that, with $U \equiv R \circ Z$,

$$\begin{bmatrix} I+U & A \circ X \\ A^* \circ X^* & I-U \end{bmatrix} \geqslant 0. \tag{21}$$

Again using Lemma 3, we can conclude from (21) that $w(A \circ X) \leqslant 1$; hence $\|S_A\|_w \leqslant 1$. $\blacksquare$

3. PROOFS OF OUR THEOREM AND THE HAAGERUP THEOREM

Proof of our theorem. $(\mathrm{i})_w$ implies $(\mathrm{ii})_w$ by Lemma 6. The equivalence of $(\mathrm{ii})_w$ and $(\mathrm{iii})_w$ is immediate by writing $B = [x_1, x_2, \dots, x_n]$. The implication $(\mathrm{ii})_w \Rightarrow (\mathrm{iv})_w$ is seen by taking $R \equiv B^*B$. In fact, $R \circ I \leqslant I$ and

$$\begin{bmatrix} R & A \\ A^* & R \end{bmatrix} = \begin{bmatrix} B^* & 0 \\ B^*W^* & B^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I - W^*W \end{bmatrix} \begin{bmatrix} B & WB \\ 0 & B \end{bmatrix} \geqslant 0.$$

Finally, the implication $(\mathrm{iv})_w \Rightarrow (\mathrm{i})_w$ follows from Lemma 7. ∎

Turning to the Haagerup theorem, remark first that the equivalence of (ii), (iii), and (iv) as well as the implication (iv) ⇒ (i) is found in [6] and is shown just as in the proof of our theorem. Haagerup's original proof of the implication (i) ⇒ (iv) seems never to have been published.

For a proof of this implication, we need one more lemma, of independent interest.

LEMMA 8.

$$\|S_A\|_\infty = \left\| S_{\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}} \right\|_w \qquad (A \in \mathbb{M}_n),$$

where 0 *is the* n*-by-*n *zero matrix.*

Proof. Remark first that by (1), for a $2n$-by-$2n$ block matrix $\begin{bmatrix} B & D \\ C & E \end{bmatrix}$,

$$2w\left(\begin{bmatrix} B & D \\ C & E \end{bmatrix}\right) \geqslant \left\|\begin{bmatrix} B & D \\ C & E \end{bmatrix}\right\|_\infty \geqslant \left\|\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}\right\|_\infty = \|D\|_\infty.$$

On the other hand, it is known (see Holbrook [2]) that

$$w\left(\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}\right) = \tfrac{1}{2}\|D\|_\infty.$$

Therefore we have

$$\left\| S_{\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}} \right\|_w = \sup\left\{ w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} B & D \\ C & E \end{bmatrix} \right);\ w\left(\begin{bmatrix} B & D \\ C & E \end{bmatrix} \right) \leqslant 1 \right\}$$

$$= \sup\left\{ w\left(\begin{bmatrix} 0 & A \circ D \\ 0 & 0 \end{bmatrix} \right);\ \|D\|_\infty \leqslant 2 \right\}$$

$$= \sup\left\{ \tfrac{1}{2}\|A \circ D\|_\infty;\ \|D\|_\infty \leqslant 2 \right\} = \|S_A\|_\infty. \qquad \blacksquare$$

Proof of implication (i) $\Rightarrow$ (iv) *in the Haagerup theorem.* Suppose that $\|S_A\|_\infty = 1$. Since by Lemma 8

$$\|S_A\|_\infty = \left\| S_{\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}} \right\|_w,$$

according to our theorem there are $R_{ij} \in \mathbb{M}_n$ $(i, j = 1, 2)$ such that

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \geqslant 0,$$

where $R_{11} \circ I \leqslant I$, $R_{22} \circ I \leqslant I$, and

$$\begin{bmatrix} R_{11} & R_{12} & 0 & A \\ R_{21} & R_{22} & 0 & 0 \\ 0 & 0 & R_{11} & R_{12} \\ A^* & 0 & R_{21} & R_{22} \end{bmatrix} \geqslant 0.$$

Then, with $R_1 \equiv R_{11}$ and $R_2 \equiv R_{22}$, we have (iv):

$$\begin{bmatrix} R_1 & A \\ A^* & R_2 \end{bmatrix} \geqslant 0 \quad \text{and} \quad R_1 \circ I \leqslant I, \quad R_2 \circ I \leqslant I. \qquad \blacksquare$$

4. CONSEQUENCES AND RELATED RESULTS

COROLLARY 1. $\|S_A\|_\infty \leqslant \|S_A\|_w \leqslant 2\|S_A\|_\infty$ $(A \in \mathbb{M}_n)$.

Proof. To see the left inequality, let $\|S_A\|_w = 1$. Then $(\mathrm{iv})_w$ implies (iv) with $R_1 = R_2 \equiv R$. The right inequality follows immediately from (1). ■

Johnson [4] showed the inequality

$$w(A \circ B) \leqslant 2w(A)w(B) \qquad (A, B \in \mathbb{M}_n),$$

which is equivalent to

$$\|S_A\|_w \leqslant 2w(A) \qquad (A \in \mathbb{M}_n).$$

In view of (1), the following result of Okubo [5] gives a refinement. Let us derive it from our theorem.

COROLLARY 2. $\|S_A\|_w \leqslant \|A\|_\infty$ $(A \in \mathbb{M}_n)$.

Proof. If $\|A\|_\infty = 1$, take $R = I$ in $(\mathrm{iv})_w$. ■

COROLLARY 3. *If A is Hermitian, then* $\|S_A\|_\infty = \|S_A\|_w$.

Proof. If $\|S_A\|_\infty = 1$, by the Haagerup theorem there are $0 \leqslant R_1, R_2 \in \mathbb{M}_n$ satisfying (iv). Since $A = A^*$, $R = \frac{1}{2}(R_1 + R_2)$ satisfies $(\mathrm{iv})_w$. Therefore $\|S_A\|_\infty \geqslant \|S_A\|_w$. The reverse inequality follows from Corollary 1. ■

COROLLARY 4. *If $A = [a_{ij}]$ is positive semidefinite,*

$$\|S_A\|_w = \max_i a_{ii}.$$

Proof. Since

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} \geqslant 0,$$

the inequality $\|S_A\|_w \leqslant \max_i a_{ii}$ follows from our theorem. The reverse inequality is immediate because

$$a_{ii} = w(S_A(E_{ii})) \qquad (i = 1, 2, \ldots, n). \qquad \blacksquare$$

Remark that since here $\|S_A\|_w = \|S_A\|_\infty$ by Corollary 3 and the map S_A is positive, the assertion of Corollary 4 is an immediate consequence of a general result that a positive linear map on a C^*-algebra attains its norm on the unit element (see [6, p. 11]):

$$\|S_A\|_\infty = \|S_A(I)\|_\infty = \|A \circ I\|_\infty = \max_i a_{ii}.$$

COROLLARY 5.

$$\|S_A\|_\infty = \left\| S_{\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}} \right\|_w \qquad (A \in \mathbb{M}_n).$$

Proof. Since $\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ is Hermitian, it suffices to prove

$$\|S_A\|_\infty = \left\| S_{\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}} \right\|_\infty \qquad (A \in \mathbb{M}_n),$$

which is, however, immediate from the definition of the norm by using the obvious relations

$$\left\| \begin{bmatrix} B & D \\ C & E \end{bmatrix} \right\|_\infty \geqslant \left\| \begin{bmatrix} 0 & D \\ C & 0 \end{bmatrix} \right\|_\infty = \max\{\|C\|_\infty, \|D\|_\infty\}. \qquad \blacksquare$$

COROLLARY 6. *If A is unitary, then* $\|S_A\|_\infty = \|S_A\|_w = 1$.

Proof. The inequality $\|S_A\|_w \leqslant 1$ follows from Corollary 2. On the other hand, since the unitarity of A implies that the Schur product $A \circ \bar{A}$ of A and its complex conjugate $\bar{A}$ is doubly stochastic, we have $\|A \circ \bar{A}\|_\infty \geqslant 1$; hence $\|S_A\|_\infty \geqslant 1$, because $\|\bar{A}\|_\infty = \|A\|_\infty = 1$. Now the assertion follows from Corollary 1. $\blacksquare$

COROLLARY 7. *For any* $A \in \mathbb{M}_n$

$$\|S_{|A|+|A^*|}\|_w \geqslant \|S_A\|_w. \tag{22}$$

If A *is normal, that is,* $|A| = |A^*|$, *then*

$$\|S_{|A|}\|_w \geqslant \|S_A\|_w.$$

Proof. The inequality (22) follows from Corollary 4 and

$$\begin{bmatrix} |A|+|A^*| & A \\ A^* & |A|+|A^*| \end{bmatrix} \geqslant 0,$$

which is a consequence of the inequality

$$\begin{bmatrix} |A^*| & A \\ A^* & |A| \end{bmatrix} \geqslant 0. \tag{23}$$

If A is normal, (23) becomes

$$\begin{bmatrix} |A| & A \\ A^* & |A| \end{bmatrix} \geqslant 0$$

and we can take $R = |A|$ instead of $|A| + |A^*|$. ∎

In closing the paper, let us show by an example that the inequality in Corollary 1 as well as the inequality (22) is best possible.

Consider

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Since

$$A = U \cdot \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \cdot U^* \qquad \text{with unitary } U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

we have

$$w(A) = w\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right) = 1$$

and

$$\|A\|_\infty = \left\|\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right\|_\infty = 2.$$

Therefore $\|S_A\|_w \leqslant \|A\|_\infty = 2$ by Corollary 2. Since

$$S_A(A) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad w\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2,$$

we can conclude that $\|S_A\|_w = 2$. Since, with $V \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$S_A(X) = VX \qquad (X \in \mathbb{M}_2),$$

we have $\|S_A\|_\infty = 1$. Further, it is easy to see that

$$|A| = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad |A^*| = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};$$

hence

$$\|S_{|A|+|A^*|}\|_w = \|S_{2I}\|_w = 2 = \|S_A\|_w = 2\|S_A\|_\infty.$$

Therefore the inequality in Corollary 1 and the inequality (22) are best possible.

It would be pleasant to be able to write condition $(\mathrm{ii})_w$ of our theorem as

$(\mathrm{ii})'_w$ *A admits a factorization* $A = B^*C$ *such that*

$$B^*B = C^*C \quad \textit{and} \quad B^*B \circ I \leqslant I,$$

since it would then be parallel to (ii) of the Haagerup theorem. This alternative formulation would be correct if the contraction W in $(\mathrm{ii})_w$ could be chosen to be unitary. Y. Nakamura has shown that this is *not* the case, so the alternative formulation (ii)$'_w$ is not correct. Nakamura suggested considering the matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Since $\|S_A\|_w = 1$ is shown above, if (ii)$'_w$ holds, then A should admit a factorization $A = B^*C$ such that $B^*B = C^*C$ and $B^*B \circ I \leqslant I$. Remark here that at least one of the diagonal entries of B^*B is equal to one and the other is not greater than one. Then a calculation, based on the positivity

$$\begin{bmatrix} B^*B & A \\ A^* & B^*B \end{bmatrix} \geqslant 0,$$

will show that B^*B must be a nonsingular diagonal matrix. Then both B and C are nonsingular, and so is B^*C. This causes a contradiction because A is singular.

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